Impulse given to a plate by a quantized vortex ring

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It is shown that a quantized vortex ring in superfluid He^4, moving freely towards a flat plate, gives to it an impulse equal to the fluid impulse of the ring, provided only that the radius of the plate is larger than the initial vortex radius but smaller than the distance to the source of the ring. This is in agreement with experiment, in contradiction to two previous theoretical analyses.

There has been some discussion recently on the question of whether a vortex ring in superfluid helium 4, moving freely towards a wall (i.e., with no forces acting on it which are nonconservative in the fluid-dynamical sense of Huggins[1]), gives to the wall a momentum impulse equal to the impulse of the vortex, $I=\pi\rho\kappa r_0^2$ (where r_0 is the radius, κ is the circulation of the vortex ring, and ρ is the fluid density). This is interesting since such an impulse would provide a method of detecting uncharged vortex rings, as suggested by Gamota and Barmatz[2] (GB), and also because it may provide some insight into the relationship between the impulse I and a momentum. The impulse I is not the additional momentum in the fluid due to the presence of a vortex ring—for an incompressible fluid this is zero in a bounded volume and in an infinite volume is undefined as the evaluation of the integrals involved depends on how infinity is approached. Thus it is not immediately obvious that there will be a momentum impulse given to the wall.

Gamota and Barmatz[2] performed an experiment in which they measured the force exerted by a beam of vortex rings on a flexible diaphragm, and their results were consistent with each vortex ring delivering its impulse I to the detector. However, two theoretical papers[3,4] showed that there could be no impulse given to the whole wall by a freely moving vortex, and suggested the force measured must be attributed to other causes.

Fetter,[3] by directly calculating the fluid motion and then using Bernoulli's theorem, calculates the force exerted at any moment on a plate forming part of an infinite plane wall due to a vortex moving towards it, and shows that the force tends to zero as the radius of the plate, R, becomes large (essentially if $R^2 \gg z^2+r_0^2$, where r_0 is the radius of the ring and z is the distance to the wall), and that the force on a finite plate is proportional to r_0. Since GB measured a force proportional to r_0^2, Fetter is led to look for some other reason for the force. But it is important to be clear that we are interested in the impulse, obtained by integrating the force on the wall over the time of the motion, and not in the force at any one moment. In the experiment, the force F due to a *beam* of vortices was measured. This is related to the impulse I_p that would be given to the wall by a single vortex over the whole of its motion by $F=I_p\times i$, where i is the number current in the beam. Fetter fails to make this point clear. His first result tells us that for motion over a finite time the impulse on an infinite plane wall is zero, but otherwise the results are not directly applicable to this problem.

Huggins[4] emphasized that there can be no momentum in the flow of an incompressible fluid in a bounded volume, and that therefore there can be no net impulse on the container walls from a freely moving vortex ring. He suggests that the force measured in the GB experiment must come from the acceleration of the rings by the electric field acting on their charges. However, it will be shown that such a general argument cannot adequately describe the experiment. Also, arguments based on the total fluid momentum are often not very useful in describing the internal motion of a fluid. For example, it is assumed that a phonon or roton of wave vector $\vec{k}$ has a momentum $\hbar\vec{k}$, for they form the normal fluid (of the two-fluid model) which has momentum density (normal density times normal velocity) correctly given by summing these "momenta." The normal fluid interacts with the wall to give the viscous effects and a force on the wall. But $\hbar\vec{k}$ is not the true momentum of a wave packet of these quasiparticles (normalized to one quasiparticle): We must associate with the wave packet a long-range backflow, so that the total fluid momentum is given by the product of the extra mass of the wave packet and its velocity, as required by the conservation of current. Perhaps for vortices as well it is useful to consider them to have a momentum together with a canceling backflow to satisfy the conservation laws. From the velocity field of the vortex, such a separation into long and short range is not obvious, but we will see that for the impulse on the wall it does seem natural.

The philosophy of calculating the impulse is the same as in Ref. 3: It is supposed that the vortices behave as their classical counterparts in an ideal fluid (as demonstrated by Rayfield and Reif[5]) with a core size a of atomic dimensions. The pressure at the wall is obtained using Bernoulli's equation in the form

$$p = p_0 + \rho \frac{\partial \phi}{\partial t} - \tfrac{1}{2}\rho v^2 , \qquad (1)$$

where p_0 is the constant pressure independent of the presence of the vortex and ϕ and $\vec{v}$ are the velocity potential and velocity ($\vec{v} = -\vec{\nabla}\phi$) at the wall. The method of images is used to satisfy the boundary condition of the zero perpendicular component of velocity at the wall. If the temperature is not absolute zero, ρ should be replaced by the superfluid density ρ_S, and the dissipation of the energy of the vortex by the normal fluid should be taken into account. However, below about 0.5°K these effects are negligible.

It is convenient to divide the motion into three parts:

(I) From the starting position z_i to a distance of about $10r_0$ from the wall. Here the effect of the wall is negligible and the vortex ring has constant radius r_0 and constant velocity. In addition, the $\frac{1}{2}\rho v^2$ term in Eq. (1) is negligible compared with the $\rho\, \partial\phi/\partial t$ term.

(II) Motion from $10r_0$ to about $1000a$ from the wall. Here the radius of the ring is no longer constant and the $\frac{1}{2}\rho v^2$ term is no longer negligible, but it seems reasonable to suppose that the semiclassical approach of the model is still a good approximation. Calculation shows that the radius of the ring increases by less than a factor of 2.

(III) The motion from $1000a$ to the wall. In this part, the classical fluid-dynamic approach must break down. To determine how the vortex ring finally disappears, the detailed behavior of liquid helium near the surface must be used, and surface inhomogeneities are probably important. Nothing is known about this part of the motion, but as in Ref. 4 it will be supposed that there is no impulse on the wall from this part.

From the work of Fetter,[3] it is immediately obvious that the impulse from region II is negligible for a large enough plate. He calculates the force from a vortex ring to go to zero as the radius of the plate becomes larger, and, since the velocity of the ring normal to the wall decreases only by a factor of about 2, the impulse from this region can be made negligible. Detailed calculation shows that the relevant criterion is that the radius of the plate is larger than a few times r_0.

The impulse from part I of the motion is very easily found: For this part, the pressure at the wall is $\rho\, \partial\phi/\partial t$ and so the momentum impulse given to the plate is

$$\rho \int dA \int dt \frac{\partial \phi}{\partial t} = \rho \int dA\, (\phi_f - \phi_i) , \qquad (2)$$

where dA is an element of area of the wall and ϕ_i and ϕ_f are the potentials at this element when the vortex is at z_i and the final position z_f, respectively. Equation (2) can be evaluated in many ways. Easiest is probably to remember that the velocity field of a vortex ring is the same as the magnetic field due to a current loop, with the equivalences: $B \equiv v$, $\phi_B \equiv \phi$, and $\mu_0 J \equiv \kappa$ (B, ϕ_B, and J are the magnetic field, potential, and the current, respectively, in the Standard International system). We immediately see that if the source distance z_i is much larger than the plate radius R, then ϕ_i and $\int \phi_i\, dA$ are negligible. The pressure impulse ($\int p\, dt$) is therefore $\rho\phi_f$, and is a sharply localized function—large only over a radius of order z_f, which is typically 100 μm. The general result for the pressure impulse at radius r from the axis of the ring is

$$\frac{I}{2\pi}\left(\frac{z_f}{(z_f^2 + r^2)^{3/2}} - \frac{z_i}{(z_i^2 + r^2)^{3/2}}\right) . \qquad (3)$$

The first term gives a pressure impulse localized over a small area of order the area of the vortex ring and integrating to give a momentum impulse equal to the impulse of the vortex, I; the second term gives a pressure impulse of opposite sign spread over a much larger area (of order z_i^2) and which gives an equal and opposite momentum impulse on integrating over an infinite wall. Therefore, Eq. (3) shows us that, at least in considering the impulse given to a wall, it does seem useful to consider the fluid flow as a region of momentum I, giving the short-range component in (3), together with a long-range backflow of opposite total momentum.

Integrating Eq. (3) over a finite plate, it follows that if an experiment could measure the momentum impulse given to a plate of radius R by a single vortex, then the result would be I if the conditions $z_i \gg R \gg r_0$ were satisfied—and this should be easy to arrange. Any actual experiment must probably measure the force due to a beam of vortex rings, and this, if the same conditions are satisfied, will be $i \times I$—and this is on the assumption that there is no impulse on the plate due to the final disappearance of the ring. These results do not depend on there being a charge on the ring, and so, in agreement with GB, it is suggested that it should be possible to detect a beam of uncharged vortex rings by measuring the force on a flexible diaphragm.

The above method of calculation is very simple, but it may be useful to explain how these results tie in with Fetter's results that the force on a large plate from a single vortex is zero and that the vortex has negligible effect on the wall if it is far enough away. The force F on a circular plate of radius R (and centered on the axis of the ring) due to a vortex distance z away (in region I of the motion) is

$$F=\rho\kappa\pi r_0^2|\dot{z}|R^2/(z^2+R^2)^{3/2}.$$

Although the force decreases with increasing radius R (and as R^{-1} for R much greater than z), the range of z over which the force is appreciable increases proportional to R. On integrating F to obtain the impulse, we are effectively integrating over a range of z proportional to R, and so the final result is independent of R. This is only true if the range of integration includes all z for which F is large, that is if $z_i \gg R$. Compare this with region II, where the range of integration is independent of R, and the impulse therefore decreases as R^{-1} and can be made negligible.

These results also enable us to analyze the GB experiment in detail using Eq. (2) for the impulse measured per vortex. This result is general—it only depends on the $\frac{1}{2}\rho v^2$ term in Eq. (1) being negligible—and so also applies to the GB experiment, where an electric field is applied over part of the motion and other boundaries are present. We can rewrite $\phi_f - \phi_i$ as $[\phi(f)-\phi(2)]-[\phi(2)-\phi(i)]$, where $\phi(j)$ is the velocity potential at the wall with the vortex at position j of Fig. 1. Here i is the point of nucleation of the vortex, and is very close to the source plate. It is easy to show, by the method of images, that $\phi(2)$ and $\phi(i)$, and therefore $\phi(2)-\phi(i)$, are negligible. (They are reduced by a factor of order the ratio of the distance of i to the source plate to some apparatus dimension.) The impulse on the plate is therefore equal to that due to an uncharged vortex ring of constant radius moving over the whole distance. Also, since $\phi(i)$ is negligible and $\phi(f)$ is large only over an area of order the area of the ring (and so can be calculated as for the infinite detector), this impulse is I in either case, independent of the size of the vessel compared with the size of the plate. Thus, that the impulse measured is I is not a trivial result of the conservation of total momentum in the container, but depends crucially on the short range of the pressure impulse for a vortex striking the plate. Huggins's[4] explanation of the impulse measured depends on the radius of the detector R and the source to detector distance D being much larger than the distance to the accelerating plates d, when it would be true that the impulse on the detector from motion in the field-free region would be negligible and that the impulse due to the change in motion between the plates would be I. However, in the experiment the field-free region was about 1 cm long (and not 38 μm as assumed by Huggins) and so cannot be neglected, and d was of the same order as R and D. For the actual experimental arrangement his argument suggests only that some (unknown) fraction of the fluid impulse I would be expected on the plate, and if $d \gg R$ the argument tells us nothing. The impulse is I in either case, as was proved above.

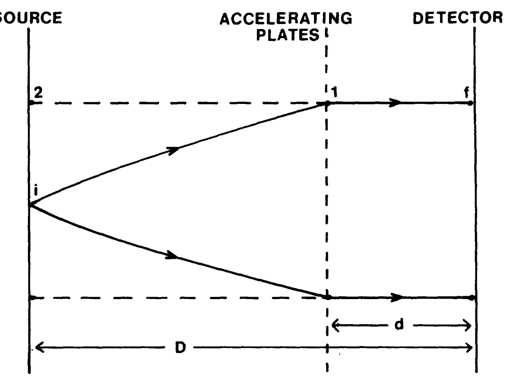

FIG. 1. Schematic diagram of the path of a vortex ring in the GB experiment. The diagram shows the path of two diametrically opposed points on the ring. The length d is about 1 cm, D about 3 cm, and the radius of the detector about 2 cm.

Finally, it is worth noting that the recent experiment by Carey, Chandrasekhar, and Dahm[6] on vortex-generated level differences in liquid helium, and in particular their conclusion that the level difference disappears when the vortex rings become smaller than the connecting orifice, gives direct experimental evidence for the short-range component of the pressure impulse due to a vortex ring.

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[1]E. R. Huggins, Phys. Rev. A 1, 327 (1970).
[2]G. Gamota and M. Barmatz, Phys. Rev. Lett. 22, 874 (1969).
[3]A. L. Fetter, Phys. Rev. A 6, 402 (1972).
[4]E. R. Huggins, Phys. Rev. Lett. 29, 1067 (1972).
[5]G. W. Rayfield and F. Reif, Phys. Rev. 136, A1194 (1964).
[6]R. Carey, B. S. Chandrasekhar, and A. J. Dahm, Phys. Rev. Lett. 31, 873 (1973).